1. Exercises from 4.1

This tutorial will focus on some aspects of the theory of Riemann integrals. Let's first remind ourselves of the relevant definitions.

- A *partition* of an interval $[a, b]$ is a collection of numbers $P = \{a = x_0 < \cdots < x_j < \cdots < x_n = b\}$.
- If P and P' are partitions of $[a, b]$ and $P \subseteq P'$, then we say that P' is a *refinement* of P.
- Let $f : [a, b] \to \mathbb{R}$ be bounded, and set

$$M_j = \sup \{f(x) : x_{j-1} \le x \le x_j\}$$

$$m_j = \inf \{f(x) : x_{j-1} \le x \le x_j\}$$

  The *upper Riemann sum*, $S_P f$, corresponding to the partition P is given by:

$$S_P f = \sum_{j=1}^{|P|} M_j (x_j - x_{j-1})$$

  And similarly for the lower Riemann sum, denoted $s_P f$.
- We say that a bounded function f is *Riemann integrable* if and only if $\inf_P S_P f = \sup_P s_P f$

The statement of the definition of integrability implicitly uses the $\forall$ quantifier. To say whether a function is integrable, you need to look at the upper and lower Riemann sums of *every* possible partition of an interval, and then check that the upper bound for the lower sums agrees with the lower bound for the upper sums. Such definitions are often computationally useless to check and we should try our best to replace them with a $\exists$ quantifier. This is the content of Folland lemma 4.5:

LEMMA. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function, then f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$, $\exists$ a partition P of $[a, b]$ such that $S_P f - s_P f < \epsilon$.*

PROOF. (Forward direction):

- Fix $\epsilon > 0$.
- Pick partitions Q and Q' such that (1) $S_Q f - \inf_P S_P f < \epsilon/2$ and (2) $\sup_P s_P f - s_{Q'} f < \epsilon/2$; these partitions exist, by the properties of inf and sup.
- Since f is integrable, we have that:

$$\sup_P s_P f = \inf_P S_P f$$

- This gives $S_Q f - s_{Q'} f < \inf_P S_P f + \epsilon/2 - \sup_P s_P f + \epsilon/2 = \epsilon$
- Now define $P = Q \cup Q'$; P is a refinement of Q and Q', so by lemma 4.3 we have the chain of inequalities:

$$s_{Q'} f \le s_P f \le S_P f \le S_Q f$$

  So $S_P f - s_P f \le S_Q f - s_{Q'} f < \epsilon$.

(Backward direction)

- Suppose that for every $\epsilon > 0$, we can find a partition Q of $[a, b]$ such that $S_Q f - s_Q f < \epsilon$
- Recall that for any partition Q,

$$\inf_P S_P f \le S_Q f$$

$$\sup_P s_P f \ge s_Q f \implies -\sup_P s_P f \le -s_Q f$$

- Therefore for every $\epsilon > 0$ we have,

$$\inf_P S_P f - \sup_P s_P f \le S_Q f - s_Q f < \epsilon$$

  implying that f is integrable.

$\square$

Let's show an example of an integrable function and calculate its integral.

PROBLEM 1. *Show that the "floor" function $f(x) = \lfloor x \rfloor$ is integrable on $[0, 2]$ and calculate its integral.*

PROOF. The idea here is to use the lemma, and show that the upper and lower sums converge to each other for appropriately chosen partitions.

- Draw a picture of the graph of $f(x)$
- Fix $\epsilon > 0$, then set $P_\epsilon = \{0, 1 - \frac{\epsilon}{3}, 1 + \frac{\epsilon}{3}, 2 - \frac{\epsilon}{3}, 2\}$
- The upper sum is given by:

$$S_{P_\epsilon} f = (0)\left(1 - \frac{\epsilon}{3}\right) + (1)\left(1 + \frac{\epsilon}{3} - 1 + \frac{\epsilon}{3}\right) + (1)\left(2 - \frac{\epsilon}{3} - 1 - \frac{\epsilon}{3}\right) + (2)\left(2 - 2 + \frac{\epsilon}{3}\right) = 1 + \frac{2\epsilon}{3}$$

and the lower sum by:

$$s_{P_\epsilon} f = (0)\left(1 - \frac{\epsilon}{3}\right) + (0)\left(1 + \frac{\epsilon}{3} - 1 + \frac{\epsilon}{3}\right) + (1)\left(2 - \frac{\epsilon}{3} - 1 - \frac{\epsilon}{3}\right) + (1)\left(2 - 2 + \frac{\epsilon}{3}\right) = 1 - \frac{\epsilon}{3}$$

- So $S_{P_\epsilon} f - s_{P_\epsilon} f = \epsilon \to 0$ since we can pick ϵ as small as we like; therefore f is integrable by the lemma.
- To calculate the integral, we simply take $\epsilon \to 0$ and notice that:

$$\int_a^b f(x)\, dx = \lim_{\epsilon \to 0} S_{P_\epsilon} f = 1$$

$\square$

The next problem is the standard example of a function which is not Riemann integrable. If someone asked you, "Give me an example of a non-integrable function", then this should always be your first thought.

PROBLEM 2. *(Folland 4.1.1)*

Let $f : \mathbb{R} \to \mathbb{R}$ be given as follows:

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} \backslash \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$$

and let $[a, b]$ be any closed interval in $\mathbb{R}$.

- Let P be any partition of $[a, b]$

$$M_j = \sup\{f(x) : x_{j-1} \le x \le x_j\} = 1\, \forall j \quad \text{since any interval } x_{j-1} < x < x_j \text{ contains a rational}$$

$$m_j = \inf\{f(x) : x_{j-1} \le x \le x_j\} = 0\, \forall j \quad \text{since any interval } x_{j-1} < x < x_j \text{ contains an irrational}$$

- Computing the upper and lower Riemann sums:

$$S_P f = \sum_{j=1}^{|P|} M_j (x_j - x_{j-1}) = \sum_{j=1}^{|P|} (x_j - x_{j-1}) = b - a \quad \text{(telescoping sum)}$$

$$s_P f = \sum_{j=1}^{|P|} m_j (x_j - x_{j-1}) = 0$$

- Therefore since the Riemann sums do not depend on the partition P, we have:

$$\inf_P S_P f = b - a$$

$$\sup_P s_P f = 0$$

Which are not equal, therefore the Riemann integral of $f(x)$ does not exist on any interval.